Algorithmic Thinking
Luay Nakhleh

Functions

Definition 1 *Let A and B be nonempty sets. A function from A to B is an assignment of exactly one element of B to each element of A. We write $f(a) = b$ if b is the unique element of B assigned by the function f to the element a of A. If f is a function from A to B, we write $f : A \to B$.*

Examples:

- $f_1 : \mathbb{N} \to \mathbb{N}$ defined by $f_1(n) = 2n$.
- Suppose we have a graph $g = (V, E)$. $f_2 : V \to \{1, 2, \ldots k\}$ defined by $f_2(u) \in \{1, 2, \ldots, k\}$ for every $u \in V$ and $f_2(u) \neq f_2(v)$ whenever $\{u, v\} \in E$.

Definition 2 *For function $f : A \to B$, we say that A is the domain of f and B is the codomain of f. If $f(a) = b$, we say that b is the image of a and a is the preimage of b. The range of f is the set of all images of elements of A. In other words, the range of f is the set*

$$\{b \in B : \exists a \in A, f(a) = b\}.$$

For example, for function f_1 above, the domain and codomain of f_1 is the set of all natural numbers ($\mathbb{N}$). The range of f_1 is the set of even positive integers (as no number in $\mathbb{N}$ can be mapped to an odd number under f_1).

Definition 3 *A function f is said to be one-to-one (sometimes written 1-1), or injective, if and only if $f(a) \neq f(b)$ whenever $a \neq b$.*

Examples:

- $f_3 : \mathbb{Z} \to \mathbb{Z}$ defined by $f_3(a) = a + 1$. f_3 is one-to-one (why?).
- $f_4 : \mathbb{Z} \to \mathbb{Z}$ defined by $f_4(a) = a^2$. f_4 is not one-to-one (why?).

Definition 4 *A function $f : A \to B$ is called onto, or surjective, if and only if for every element $b \in B$, there is an element $a \in A$ with $f(a) = b$.*

For example, the function f_3 above is onto (why?), whereas the function f_4 is not onto (why?).

Definition 5 *A function f is a bijection if it is both one-to-one and onto.*

For example, for a set A, the identity function

$$f(a) = a, \quad \forall a \in A$$

is a bijection.

A very important result in math is the following.

Theorem 1 *Let A and B be two sets. If there exists a bijection $f : A \to B$, then $|A| = |B|$.*

Using this result, we can prove that two sets are of the same cardinality by showing a bijection from one to the other.

Definition 6 *Let $f : A \to B$ be a bijection. The inverse function of f, dented by f^{-1}, is given by $f^{-1}(b) = a$ when $f(a) = b$.*

For example, for f_3 above, the inverse function f_3^{-1} is given by

$$f_3^{-1}(x) = x - 1.$$

Definition 7 *Let $g : A \to B$ and $f : B \to C$ be two functions. The composition of f and g, denoted by $f \circ g$, is define by*

$$(f \circ g)(x) = f(g(x)).$$

For example, let f_5 and f_6 be two functions from the set of integers to the set of integers defined by

$$f_5(x) = 2x + 3 \qquad and \qquad f_6(x) = 3x + 2.$$

Then, we have

$$(f_5 \circ f_6)(x) = f_5(f_6(x)) = f_5(3x + 2) = 2(3x + 2) + 3 = 6x + 7$$

and

$$(f_6 \circ f_5)(x) = f_6(f_5(x)) = 6x + 11.$$

Remark. A function $f : A \to B$ can also be defined in terms of a relation from A to B. A relation from A to B is just a subset of $A \times B$. A relation from A to B that contains one, and only one, ordered pair (a, b) for every element $a \in A$, defined a function f from A to B. For example, the function $f_7 : \{a, b, c\} \to \{0, 1\}$ defined by

$$f_7(a) = 0 \qquad f_7(b) = 1 \qquad f_7(c) = 0$$

can be represented as the relation

$$\{(a, 0), (b, 1), (c, 0)\}.$$